LESSON 4.1

PLANNING

LESSON OUTLINE

One day:

20 min Investigation

10 min Sharing

5 min Closing

10 min Exercises

MATERIALS

- construction tools
- protractors
- scissors
- Triangle Sum Conjecture (W), *optional*
- Sketchpad demonstration The Triangle Sum, *optional*
- Sketchpad activity Triangle Sum Conjecture, *optional*

TEACHING

Students think about the sum of the measures of the angles of a triangle, some explanations, and some consequences.

One Step

Pose this problem: "Draw a triangle on one patty paper. Create a second triangle on another patty paper by tracing two of the angles of the original triangle but making the side between the two angles longer. Guess how much larger the third angle of the new triangle is than the third angle of the original triangle. Justify your conjecture." Because the longer sides may make the angles on the second triangle look larger, students may be surprised to find that the angle measures are about the same. If students complain that the problem is misleading, remind them that real-world problems often make hidden assumptions, so students need to be skeptical.

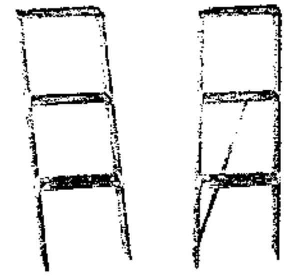
LESSON 4.1

Teaching is the art of assisting discovery.

ALBERT VAN DOREN

Triangle Sum Conjecture

Triangles have certain properties that make them useful in all kinds of structures, from bridges to high-rise buildings. One such property of triangles is their rigidity. If you build shelves like the first set shown at right, they will sway. But if you nail another board at the diagonal as in the second set, creating a triangle, you will have rigid shelves.



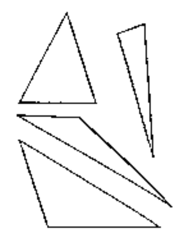
Another application of triangles is a procedure used in surveying called **triangulation**. This procedure allows surveyors to locate points or positions on a map by measuring angles and distances and creating a network of triangles. Triangulation is based on an important property of plane geometry that you will discover in this lesson.

Investigation
The Triangle Sum

You will need

- a protractor
- a straightedge
- scissors
- patty paper

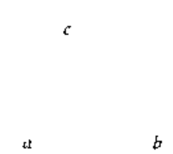
There are an endless variety of triangles that you can draw, with different shapes and angle measures. Do their angle measures have anything in common? Start by drawing different kinds of triangles. Make sure your group has at least one acute and one obtuse triangle.



Step 1 Measure the three angles of each triangle as accurately as possible with your protractor.

Step 2 Find the sum of the measures of the three angles in each triangle. Compare results with others in your group. Does everyone get about the same result? What is it? 180°

Step 3 Check the sum another way. Write the letters *a*, *b*, and *c* in the interiors of the three angles of one of the triangles, and carefully cut out the triangle.



During class work and Sharing, keep asking questions to urge a full justification, namely, the Triangle Sum Conjecture.

You might demonstrate the investigation as a follow-along activity by cutting out the triangles and displaying and measuring the pieces on an overhead projector.

Step 1 The triangles students measure should be large to reduce error. To save time, you might ask each group to explore a different triangle (obtuse, right, or acute). Each student measures the angles and the group averages the results. Then groups combine their findings. The technique of splitting up tasks and sharing results (cooperative group jigsaw method) is useful for saving time when gathering a lot of data.

Step 2 Some individuals, or even groups, may not get close enough to 180° to see a pattern. Don't press for that result at this stage.

Step 4 Tear off the three angles. Arrange them so that their vertices meet at a point. How does this arrangement show the sum of the angle measures?

Step 5 Compare results with others in your group. State your observations as a conjecture.

Triangle Sum Conjecture

C-17

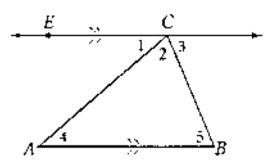
The sum of the measures of the angles in every triangle is _?_. 180°

Developing Proof The investigation may have convinced you that the Triangle Sum Conjecture is true, but can you explain *why* it is true for every triangle?

As a group, explain why the Triangle Sum Conjecture is true by writing a **paragraph proof**, a deductive argument that uses written sentences to support its claims with reasons.

Another reasoning strategy you might use is to add an **auxiliary line**, an extra line or segment that helps with a proof. Your group may have formed an auxiliary line by rearranging the angles in the investigation. If you rotated $\angle A$ and $\angle B$ and left $\angle C$ pointing up, then how is the resulting line related to the original triangle? Draw any $\triangle ABC$ and draw in that auxiliary line.

The figure at right includes $\overleftrightarrow{EC}$, an auxiliary line parallel to side $\overline{AB}$. Use this diagram to discuss these questions with your group.



- What are you trying to prove?
- What is the relationship among $\angle 1$, $\angle 2$, and $\angle 3$?
- Why was the auxiliary line drawn to be parallel to one of the sides?
- What other congruencies can you determine from the diagram?

Use your responses to these questions to mark your diagram. Discuss how you can use the information you have to prove that the Triangle Sum Conjecture is true for every triangle. As a group, write a paragraph proof. When you are satisfied with your group's proof, compare it to the one presented on the next page.

Guiding the Investigation (continued)

Steps 3, 4 You might have students tear off two of the three angles from the triangle. Then they can glue the remaining part of the triangle onto a piece of paper and line up the two angles adjacent to the third angle to notice that the three angles form a straight line. This can also be done quickly with patty paper by tracing each angle in turn, making sure each angle is adjacent to the last.

Step 5 One conjecture students might make is that the sum of the measures is larger than the measure of each angle. Another is that the sum of the measures is a straight line.

Developing Proof

If some groups finish early, [Ask] "What conjectures did you use as the basis for your proof?" Students will answer this question in Exercise 15.

Make sure that students don't rush through these questions. If they reason through the answers carefully and mark their diagram, they will have most of the proof completed. They will also have a foundation to better understand the proof given on the next page.

The reasoning strategy of adding an auxiliary line is an important tool for completing many geometric proofs. Here the arrangement of the three torn angles from the investigation gives a visual clue about where to add an auxiliary line for the proof.

Challenge students who finish early to find other proofs explaining the Triangle Sum Conjecture.

NCTM STANDARDS

CONTENT		PROCESS	
	Number		Problem Solving
✓	Algebra	✓	Reasoning
✓	Geometry	✓	Communication
✓	Measurement	✓	Connections
✓	Data/Probability	✓	Representation

LESSON OBJECTIVES

- Discover and explain the sum of the measures of the angles of a triangle
- Develop inductive and deductive reasoning
- Practice using geometry tools

EXAMPLE

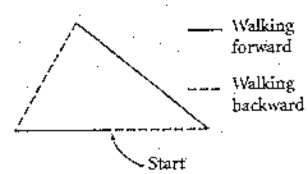
[Alert] Students may miss the fact that only the third angle of the triangle is being constructed, not the entire triangle. You might take advantage of any such misunderstanding to foreshadow the determining properties of triangles that are studied later in the chapter. [Ask] "Is knowing three angles of a triangle enough to determine the triangle?" [no] "What else must you know?" You need not answer this question now. Students will use this construction in Exercises 10 and 12 in this lesson, as well as in their investigation of the Side-Angle-Angle congruence shortcut in Lesson 4.5.

SHARING IDEAS

Have students share a variety of ideas about their conjectures and proofs. Ask for critiques of the conjecture statements; for example, if one incomplete conjecture is that the sum of the measures is a straight line, elicit the point that a sum is a number and a straight line isn't.

[Ask] "Which did you find more convincing—the investigation or the proof?" Allow students to express their opinions without judgment. Some students may note that angles 1, 2, and 3 in the diagram for the proof resemble the three torn angles meeting at a point in Step 4 of the investigation.

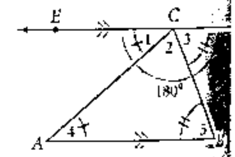
For the Triangle Sum Conjecture, a student (or you) might offer an explanation that involves imagining walking clockwise around the triangle. If you start somewhere along one edge, you rotate clockwise at each vertex and end up back where you started, so the sum of the angle measures must be 360°. Encourage creative thinking such as this, which can give insight into other polygons later in the course. [Ask] "Why is the result not 180°?" As you ask for clarification, have the students draw in the angles through which the walker turns. (Or a student might go through the motions, following a triangle of masking tape fixed to the floor.) The turning angles won't be the angles of the triangle; instead, the turning angles provide an explanation of the Exterior Angle Sum Conjecture. However, if the walker turns through the interior angles, he or she will be walking backward along one side and along part of the side where the walk began. Ask students to model such a walk and to explain the result in light of the Triangle Sum Conjecture. [When back at the starting point, the walker is facing in the opposite direction, having turned through 180°.]



Paragraph Proof: The Triangle Sum Conjecture

Consider ∠1 and ∠2 together as a single angle that forms a linear pair with ∠3. By the Linear Pair Conjecture, their measures must add up to 180°.

$m\angle 1 + m\angle 2 + m\angle 3 = 180°$



$\overline{AC}$ and $\overline{BC}$ form transversals between parallel lines $\overleftrightarrow{EC}$ and $\overleftrightarrow{AB}$. By the Alternate Interior Angles Conjecture, ∠1 and ∠4 are congruent and ∠3 and ∠5 are congruent, so their measures are equal.

$m\angle 1 = m\angle 4$

$m\angle 3 = m\angle 5$

Substitute $m\angle 4$ for $m\angle 1$, and $m\angle 5$ for $m\angle 3$ in the first equation above.

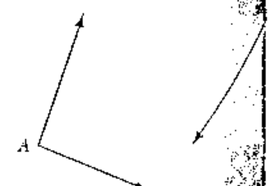
$m\angle 4 + m\angle 2 + m\angle 5 = 180°$

Therefore, the measures of the angles in a triangle add up to 180°. ■

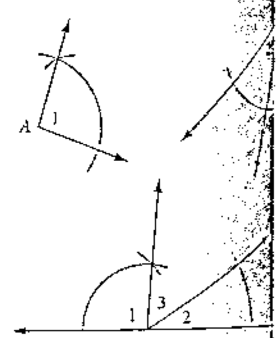
So far, you have been writing deductive arguments to explain why conjectures are true. The paragraph proof format puts a little more emphasis on justifying your reasons. You will also learn about another proof format later in this chapter.

If you have two angles of a triangle, you can use the Triangle Sum Conjecture to construct the third angle. This example shows one way to do this.

EXAMPLE Given ∠A and ∠N, construct ∠D, the third angle of △AND.



➤ Solution Label ∠A and ∠N as ∠1 and ∠2 respectively. Draw a line. Duplicate ∠1 opening to the left on this line. Duplicate ∠2 opening to the right at the same vertex on this line. Because the measures of the three angles add to 180°, the measure of ∠3 is equal to that of ∠D.



EXERCISES

You will need

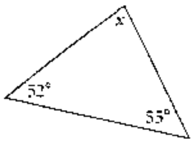
Geometry software for Exercise 1

Construction tools for Exercises 10–13

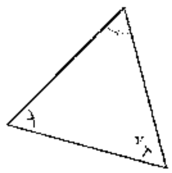
1. *Technology* Using geometry software, construct a triangle. Use the software to measure the three angles and calculate their sum. Drag the vertices and describe your observations. The angle measures change, but the sum remains 180°.

Use the Triangle Sum Conjecture to determine each lettered angle measure in Exercises 2–5. You might find it helpful to copy the diagrams so you can write on them.

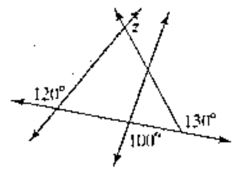
2. $x = \underline{?}$ 73°



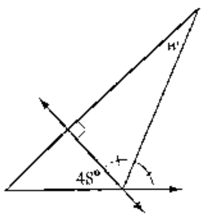
3. $v = \underline{?}$ 60°



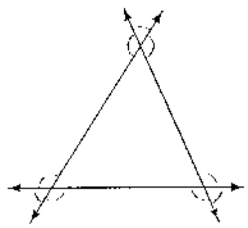
4. $z = \underline{?}$ (h) 110°



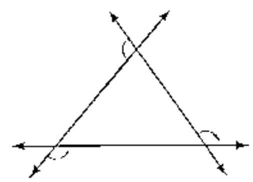
5. $w = \underline{?}$ 24°



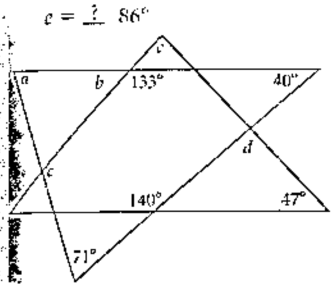
6. Find the sum of the measures of the marked angles. (h) 3 · 360° − 180° = 900°



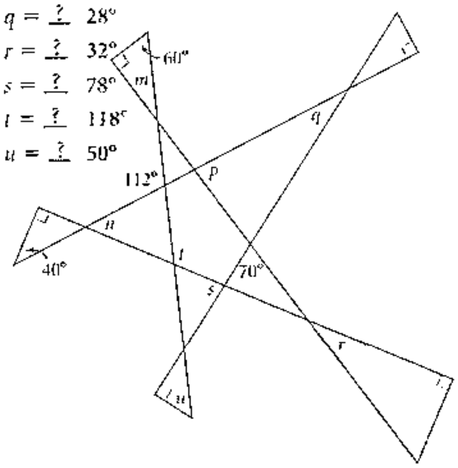
7. Find the sum of the measures of the marked angles. (h) 3 · 180° − 180° = 360°



8. $a = \underline{?}$ (h) 69°
$b = \underline{?}$ 47°
$c = \underline{?}$ 116°
$d = \underline{?}$ 93°
$e = \underline{?}$ 86°



9. $m = \underline{?}$ 30°
$n = \underline{?}$ 50°
$p = \underline{?}$ 82°
$q = \underline{?}$ 28°
$r = \underline{?}$ 32°
$s = \underline{?}$ 78°
$t = \underline{?}$ 118°
$u = \underline{?}$ 50°



Assessing Progress

Observe how well students are collecting data and making conjectures from these data. Assess their skill at using a protractor, a straightedge, and patty paper and their understanding of acute triangle, obtuse triangle, measure of an angle, and the Alternate Interior Angles Conjecture.

Closing the Lesson

Remind students of the major points of this lesson: The Triangle Sum Conjecture says that the sum of the measures of the angles of any triangle is 180°. This conjecture can be proved in a number of ways.

BUILDING UNDERSTANDING

The exercises focus on the Triangle Sum Conjecture.

ASSIGNING HOMEWORK

Essential	2–16 evens
Performance assessment	9
Portfolio	8, 19
Journal	15, 18
Group	1–13 odds
Review	20–25
Algebra review	19

MATERIALS

- Exercises 8 and 9 (T), *optional*

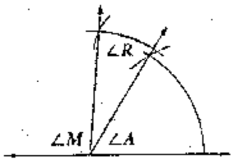
▶Helping with the Exercises

Exercises 6, 7 If students are finding these exercises difficult, suggest that they think about subtraction or that they look at the hints.

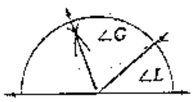
Exercises 8, 9 These exercises may be a bit intimidating at first. You might do one in class, with students working in pairs explaining how they know each answer. This is another good opportunity for think-aloud pair-share where students take turns finding an answer and then explaining it. As needed, point out that some of the angles are in both a larger and a smaller triangle. Remind students that practice makes perfect and that it is much easier to practice in pencil than in pen.

Exercise 10 This construction could also be done by constructing the given angles on the ends of a line segment; then $\angle M$ is the third angle formed where the sides of the angles meet.

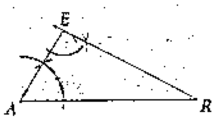
10.



11.

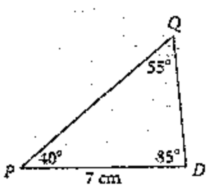


12. First construct $\angle E$, using the method used in Exercise 10.



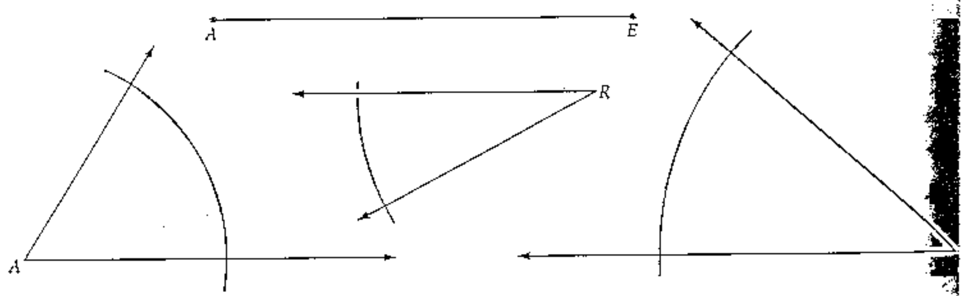
15. Answers will vary. See the proof on page 202. To prove the Triangle Sum Conjecture, the Linear Pair Conjecture and the Alternate Interior Angles Conjecture must be accepted as true.

16. It is easier to draw $\triangle PDQ$ if the Triangle Sum Conjecture is used to find that the measure of $\angle D$ is 85°. Then $\overline{PD}$ can be drawn to be 7 cm, and angles P and D can be drawn at each endpoint using the protractor.



Exercise 17 If you did the one-step investigation, students may already have discovered the Third Angle Conjecture.

In Exercises 10–12, use what you know to construct each figure. Use only a compass and a straightedge.



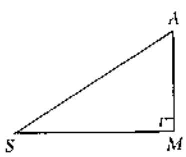
10. *Construction* Given $\angle A$ and $\angle R$ of $\triangle ARM$, construct $\angle M$.

11. *Construction* In $\triangle LEG$, $m\angle E = m\angle G$. Given $\angle L$, construct $\angle G$. ⓗ

12. *Construction* Given $\angle A$, $\angle R$, and side $\overline{AE}$, construct $\triangle EAR$. ⓗ

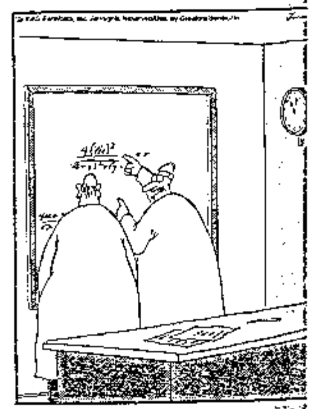
13. *Construction* Repeat Exercises 10–12 with patty-paper constructions.

14. *Developing Proof* In $\triangle MAS$ below, $\angle M$ is a right angle. Let's call the two acute angles, $\angle A$ and $\angle S$, "wrong angles." Write a paragraph proof or use algebra to show that "two wrongs make a right," at least for angles in a right triangle.



From the Triangle Sum Conjecture $m\angle A + m\angle S + m\angle M = 180°$. Because $\angle M$ is a right angle, $m\angle M = 90°$. By substitution, $m\angle A + m\angle S + 90° = 180°$. By subtraction, $m\angle A + m\angle S = 90°$. So two wrongs make a right!

THE FAR SIDE® By GARY LARSON



"Yes, yes, I *know* that, Sidney—everybody knows *that!* ... But look: Four wrongs *squared*, minus two wrongs to the fourth power, divided by this formula, *do* make a right."

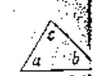
15. *Developing Proof* In your own words, prove the Triangle Sum Conjecture. What conjectures must we accept as true in order to prove it?

16. Use your ruler and protractor to draw $\triangle PDQ$ if $m\angle P = 40°$, $m\angle Q = 55°$, and $PD = 7$ cm. How can the Triangle Sum Conjecture make this easier to do?

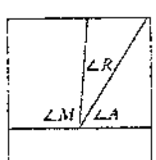
17. *Mini-Investigation* Suppose two angles of one triangle have the same measures as two angles of another triangle. What can you conclude about the third pair of angles? The third angles of the triangles also have the same measures.

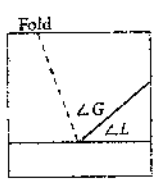
Draw a triangle on your notebook paper. Create a second triangle on patty paper by tracing two of the angles of your original triangle, but make the side between your new angles longer than the corresponding side in the original triangle. How do the third angles in the two triangles compare?

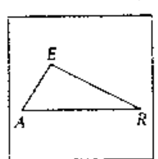
Conjecture: If two angles of one triangle are equal in measure to two angles of another triangle, then the third angles of the triangles __?__. (Third Angle Conjecture) are equal in measure



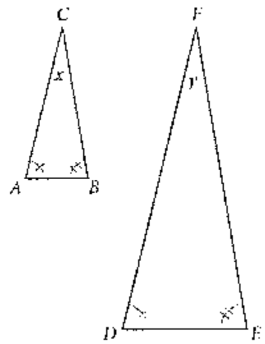
13.







18. ***Developing Proof*** Use the Triangle Sum Conjecture and the figures at right to write a paragraph proof explaining why the Third Angle Conjecture is true.



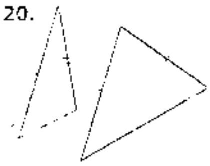
19. ***Developing Proof*** Write a paragraph proof, or use algebra, to explain why each angle of an equiangular triangle measures 60°.

For any triangle, the sum of the angle measures is 180°, by the Triangle Sum Conjecture. Since the triangle is equiangular, each angle has the same measure, say x. So $x + x + x = 180°$, and $x = 60°$.

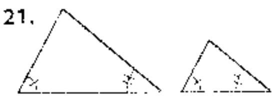
Review

In Exercises 20–24, tell whether the statement is true or false. For each false statement, explain why it is false or sketch a counterexample.

20. If two sides in one triangle are congruent to two sides in another triangle, then the two triangles are congruent. false

20. 

21. If two angles in one triangle are congruent to two angles in another triangle, then the two triangles are congruent. false

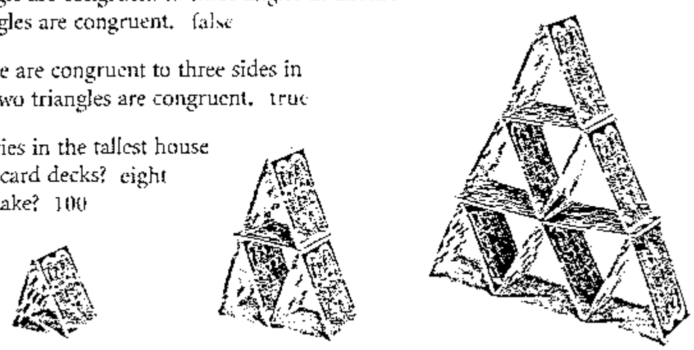
21. 

22. If a side and an angle in one triangle are congruent to a side and an angle in another triangle, then the two triangles are congruent. false

23. If three angles in one triangle are congruent to three angles in another triangle, then the two triangles are congruent. false

24. If three sides in one triangle are congruent to three sides in another triangle, then the two triangles are congruent. true

25. What is the number of stories in the tallest house you can build with two 52-card decks? eight How many cards would it take? 100

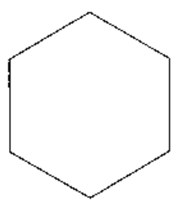


One story (2 cards) Two stories (7 cards) Three stories (15 cards)

IMPROVING YOUR VISUAL THINKING SKILLS

Dissecting a Hexagon I

Trace this regular hexagon twice.

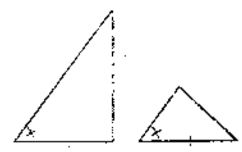


1. Divide one hexagon into four congruent trapezoids.
2. Divide the other hexagon into eight congruent parts. What shape is each part?

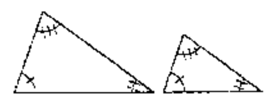
18. You know from the Triangle Sum Conjecture that $m\angle A + m\angle B + m\angle C = 180°$, and $m\angle D + m\angle E + m\angle F = 180°$. By the transitive property, $m\angle A + m\angle B + m\angle C = m\angle D + m\angle E + m\angle F$. You also know that $m\angle A = m\angle D$, and $m\angle B = m\angle E$. You can substitute for $m\angle D$ and $m\angle E$ in the longer equation to get $m\angle A + m\angle B + m\angle C = m\angle A + m\angle B + m\angle F$. Subtracting equal terms from both sides, you are left with $m\angle C = m\angle F$.

Exercises 20–24 These exercises are very important preparation for the lessons on congruence shortcuts. You might want to start some of them in class by asking for and demonstrating counterexamples. Then complete the discussion of these exercises at the start of the next class period.

22.



23.

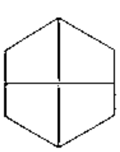


EXTENSION

Use Take Another Look activity 1 on page 255.

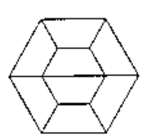
IMPROVING VISUAL THINKING SKILLS

1. If students are having difficulty, ask if they can divide the hexagon into two congruent parts.



Each part is a right trapezoid.

2. Each part is a trapezoid with three congruent sides.



LESSON 4.2

PLANNING

LESSON OUTLINE

One day:

20 min Investigation

10 min Sharing

5 min Closing

10 min Exercises

MATERIALS

- tracing paper or patty paper
- protractors
- construction tools
- Sketchpad activity Properties of Isosceles Triangles, *optional*

TEACHING

Students inductively conjecture that the base angles of an isosceles triangle are congruent, and the converse.

Start with the one-step investigation, or introduce the vocabulary before students begin to follow the steps of the investigations.

INTRODUCTION

[Ask] "What is the definition of an isosceles triangle?" [a triangle with at least two congruent sides] As needed, review vocabulary associated with an isosceles triangle: vertex angle and base angles.

[ELL] Enlarge the diagram of an isosceles triangle given on this page. Students may assume that the base must be horizontal, so you might make another diagram with the triangle in nonstandard position and prompt students to help you correctly label the sides and angles. Display the diagrams as students work on the investigations to help them use the terms correctly.

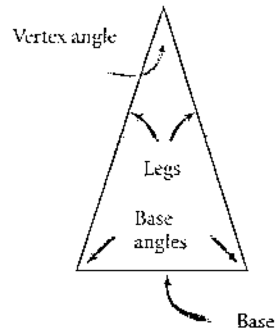
LESSON 4.2

Properties of Isosceles Triangles

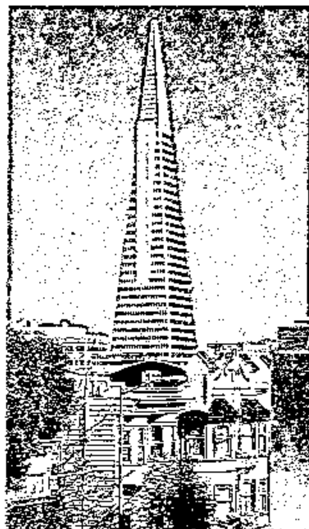
Imagination is built upon knowledge.

ELIZABETH STUART PHELPS

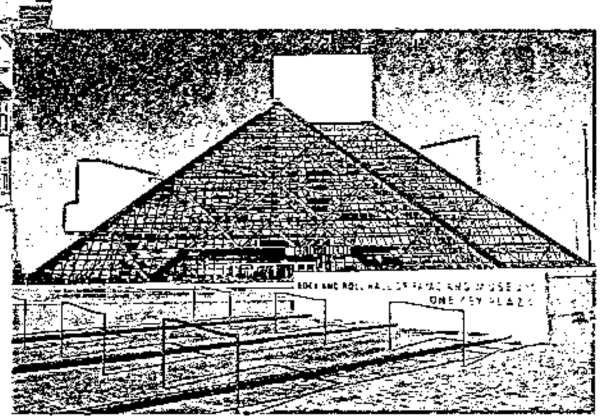
Recall from Chapter 1 that an isosceles triangle is a triangle with at least two congruent sides. In an isosceles triangle, the angle between the two congruent sides is called the vertex angle, and the other two angles are called the base angles. The side between the two base angles is called the base of the isosceles triangle. The other two sides are called the legs.



In this lesson you'll discover some properties of isosceles triangles.



The famous Transamerica Building in San Francisco contains many isosceles triangles.



The Rock and Roll Hall of Fame and Museum structure is a pyramid containing many triangles that are isosceles *and* equilateral.

Architecture CONNECTION

The Rock and Roll Hall of Fame and Museum in Cleveland, Ohio, is a dynamic structure. Its design reflects the innovative music that it honors. The front part of the museum is a large glass pyramid, divided into small triangular windows. The pyramid structure rests on a rectangular tower and a circular theater that looks like a performance drum. Architect I. M. Pei (b 1917) used geometric shapes to capture the resonance of rock and roll musical chords.

LESSON OBJECTIVES

- Discover a relationship between the base angles of an isosceles triangle
- Learn new vocabulary
- Develop problem-solving skills and inductive reasoning
- Practice using construction tools

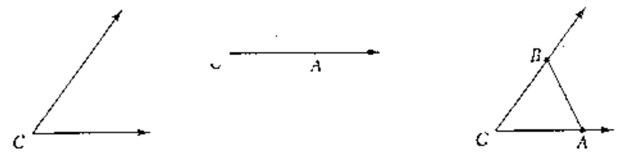
NCTM STANDARDS

CONTENT		PROCESS	
	Number	✓	Problem Solving
	Algebra	✓	Reasoning
✓	Geometry	✓	Communication
✓	Measurement	✓	Connections
	Data/Probability		Representation

Investigation 1
Base Angles in an Isosceles Triangle

You will need
- patty paper
- a straightedge
- a protractor

Let's examine the angles of an isosceles triangle. Each person in your group should draw a different angle for this investigation. Your group should have at least one acute angle and one obtuse angle.



Step 1 Step 2 Step 3

Step 1 Draw an angle on patty paper. Label it $\angle C$. This angle will be the vertex angle of your isosceles triangle.

Step 2 Place a point A on one ray. Fold your patty paper so that the two rays match up. Trace point A onto the other ray.

Step 3 Label the point on the other ray point B. Draw $\overline{AB}$. You have constructed an isosceles triangle. Explain how you know it is isosceles. Name the base and the base angles. *$\overline{CB} \cong \overline{CA}$ because they were drawn from the same point with the same distance. $\overline{AB}$, $\angle A$, $\angle B$*

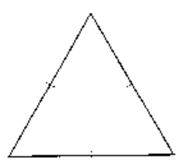
Step 4 Use your protractor to compare the measures of the base angles. What relationship do you notice? How can you fold the paper to confirm your conclusion? *$m\angle A = m\angle B$; Fold the paper so that $\angle A$ and $\angle B$ coincide.*

Step 5 Compare results in your group. Was the relationship you noticed the same for each isosceles triangle? State your observations as your next conjecture.

Isosceles Triangle Conjecture C-18

If a triangle is isosceles, then __?__ *its base angles are congruent.*

Equilateral triangles have at least two congruent sides, so they fit the definition of isosceles triangles. That means any properties you discover for isosceles triangles will also apply to equilateral triangles. How does the Isosceles Triangle Conjecture apply to equilateral triangles?



You can switch the "if" and "then" parts of the Isosceles Triangle Conjecture to obtain the converse of the conjecture. Is the converse of the Isosceles Triangle Conjecture true? Let's investigate.

One Step

Pose this problem: "Try to construct a triangle with both of these properties: The triangle has two equal sides (that is, it's isosceles), and the measures of all three angles are different." As you circulate among groups, encourage creative thinking. If students say that having two sides with equal length requires that two angles be equal, ask them to state a conjecture describing which two angles must be equal. Respond similarly if they claim that equal angles imply equal sides. Challenge them to extend their conjectures to equilateral triangles, trapezoids, and other quadrilaterals. During Sharing, remind students of the meaning of an isosceles triangle, and introduce the terminology *base angles, vertex angle,* and *equiangular triangle.* Use the terms to elicit formal statements of the Isosceles Triangle Conjecture and its converse.

Guiding Investigation 1

Step 3 [Alert] Students may have difficulty if the vertex angle is lower on their paper than the base angles. You might suggest that they rotate their paper if this is the case.

Guiding Investigation 2

Remind students of how to find a converse of a statement and that the converse of a true statement may be false.

Steps 1–4 If you used the pair-share cooperative group strategy for the first investigation, have students switch partners or roles for this investigation.

Students will prove these conjectures in Lesson 4.7, Exercises 3 and 4.

SHARING IDEAS

Have students share conjectures and come to a class consensus on their statements, to be entered into students' notebooks. As needed, discuss the converse of a conjecture. **[Ask]** "What is the converse of the converse?" [the original conjecture] A converse of a true conjecture need not be true, but if it is, the two can be combined into a biconditional "if and only if" statement. **[Language]** *Bi* means "two," as in *biweekly* and *bicycle.* The word *biconditional* means "two conditions"; either part of the conjecture can be the condition, with the other part the conclusion.

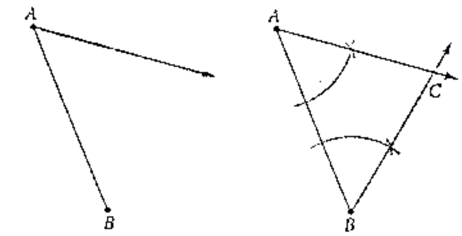
[Alert] Some students have difficulty understanding that the Isosceles Triangle Conjecture and its converse are not the same thing. Restating the conjectures in exactly opposite forms may help: If a triangle has at least two congruent sides, then the angles opposite those sides are congruent. If a triangle has at least two congruent angles, then the sides opposite those angles are congruent. **[Ask]** "What are you beginning with in each conjecture?" [The Isosceles Triangle Conjecture begins with (at least) two congruent sides, and the converse begins with two congruent angles.]

Investigation 2
Is the Converse True?

You will need

- a compass
- a straightedge

Suppose a triangle has two congruent angles. Must the triangle be isosceles?



Step 1 Step 2

Step 1 The sum of the angles of the triangle would be more than 180°.

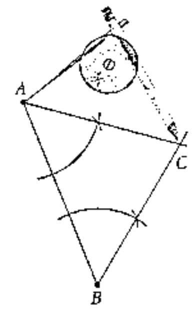
Step 1 Draw a segment and label it $\overline{AB}$. Draw an acute angle at point A. This angle will be a base angle. (Why can't you draw an obtuse angle as a base angle?)

Step 2 Copy $\angle A$ at point B on the same side of $\overline{AB}$. Label the intersection of the two rays point C.

Step 3 $AC = BC$. Fold the paper through point C so that $\angle A$ coincides with $\angle B$.

Step 3 Use your compass to compare the lengths of sides $\overline{AC}$ and $\overline{BC}$. What relationship do you notice? How can you use patty paper to confirm your conclusion?

Step 4 Compare results in your group. State your observation as your next conjecture.



Converse of the Isosceles Triangle Conjecture

If a triangle has two congruent angles, then __?__. it is an isosceles triangle

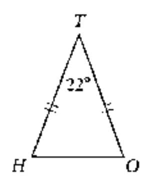
Exercises

You will need

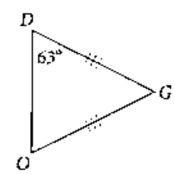
Construction tools for Exercises 15–17

For Exercises 1–6, use your new conjectures to find the missing measures.

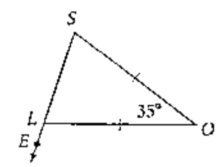
1. $m\angle H = \underline{?}$ ⓗ 79°



2. $m\angle G = \underline{?}$ 54°



3. $m\angle OLE = \underline{?}$ 107.5°

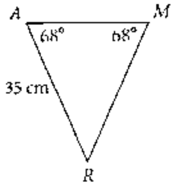


[Alert] Some students may phrase their conjectures as something like "Congruent angles make congruent sides" and vice versa. **[Ask]** "What does *make* mean?" "Do congruent angles make congruent sides in a quadrilateral?" [no]

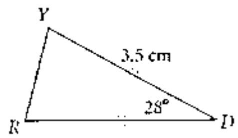
Assessing Progress

Your observations of group work and presentations give you opportunities to assess students' understanding of isosceles triangle, vertex angle, base angle, ray, acute angle, obtuse angle, measure of an angle, congruent segments, congruent angles, and equilateral, equiangular, and regular polygons. Watch for their skill in using patty paper and a straightedge to copy segments and angles and in comparing lengths with a compass. Check how well they can write the converse of a given if-then statement.

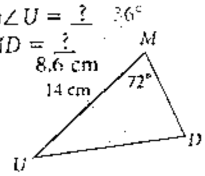
I. $m\angle R = \underline{?}$ 44°
$RM = \underline{?}$ 35 cm



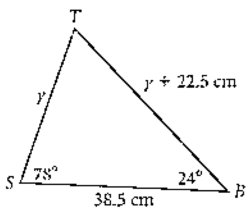
5. $m\angle Y = \underline{?}$ 76°
$RD = \underline{?}$ 3.5 cm



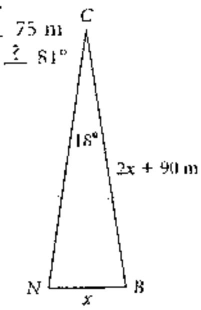
6. The perimeter of $\triangle MUD$ is 36.6 cm.
$m\angle D = \underline{?}$ 72°
$m\angle U = \underline{?}$ 36°
$MD = \underline{?}$ 8.6 cm



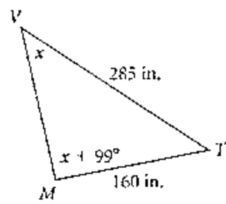
'. $m\angle T = \underline{?}$ 78°
perimeter of $\triangle TBS = \underline{?}$ 93 cm



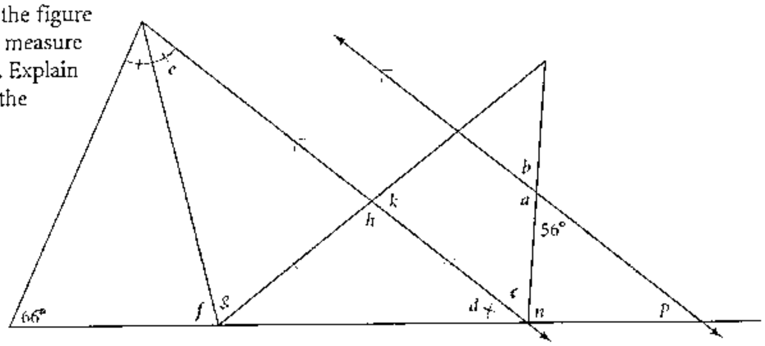
8. The perimeter of $\triangle NBC$ is 555 m.
$NB = \underline{?}$ 75 m
$m\angle N = \underline{?}$ 81°



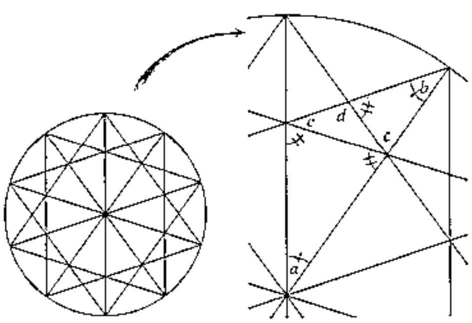
9. The perimeter of $\triangle MTV$ is 605 in.
$MV = \underline{?}$ 160 in.
$m\angle M = \underline{?}$ 126°

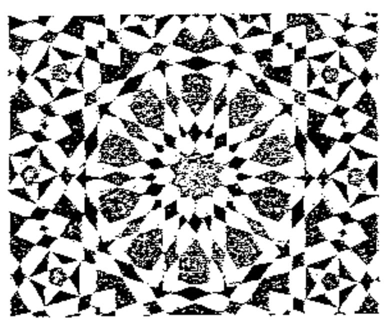


). *Developing Proof* Copy the figure at right. Calculate the measure of each lettered angle. Explain how you determined the measures d and h. ⓗ



l. The Islamic design below right is based on the star decagon construction shown below left. The ten angles surrounding the center are all congruent. Find the lettered angle measures. How many triangles are not isosceles? ⓗ $a = 36°$, $b = 36°$, $c = 72°$, $d = 108°$, $e = 36°$; none





Closing the Lesson

Remind students that the major conjectures of this lesson form a biconditional: Two sides of a triangle are congruent if and only if the angles opposite those sides are congruent. If needed, you might work one of Exercises 1–9.

BUILDING UNDERSTANDING

The exercises apply the Isosceles Triangle Conjecture and its converse.

ASSIGNING HOMEWORK

Essential	1–10
Performance assessment	11
Portfolio	6
Journal	12
Group	10
Review	13–25
Algebra review	7–9, 18–21, 24, 25

MATERIALS

- Exercises 10 and 11 (T), *optional*
- Exercises 24 and 25 (T), *optional*

▶ Helping with the Exercises

Exercises 7–9 Students will find these multistep exercises easier if they have previously worked on Exercises 4–6.

Exercise 9 If students remain focused on the goal, they need not solve for x. Once they have determined that $3x + 99 = 180$, they can divide both sides by 3 and then add 66 to get $x + 99$, which is the measure of $\angle M$. Encourage a variety of legitimate approaches.

: 10 If students are having difficulty getting l, suggest that they try working backward as ; forward: "What might I find out that would me to calculate this angle's measure?" It may :essary to find measures of angles that aren't d. Encourage students to look for larger les made up of smaller triangles. As needed, n them not to assume that n is 90°; it's not d as such.

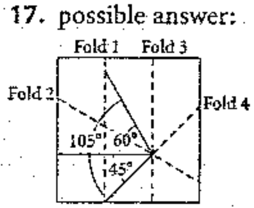
$= 124°$, $b = 56°$, $c = 56°$, $d = 38°$, $e = 38°$, $= 76°$, $g = 66°$, $h = 104°$, $k = 76°$, $n = 86°$, $p = 38°$; Possible explanation: The angles with measures 66° and d form a triangle with the angle with measure e and its adjacent angle. Because d, e, and the adjacent angle are all congruent, $3d + 66° = 180°$. Solve to get $d = 38°$. This is also the measure of one of the base angles of the isosceles triangle with vertex angle measure h. Using the Isosceles Triangle Conjecture, the other base angle measures d, so $2d + h = 180°$, or $76° + h = 180°$. Therefore, $h = 104°$.

Exercise 12 Students can use the Dynamic Geometry Exploration at www.keymath.com/DG to investigate the properties of a triangle with one vertex at the center of a circle and the other two vertices on the circumference of the circle.

12a. **Yes. Two sides are radii of a circle. Radii must be congruent; therefore, each triangle must be isosceles.**

Exercises 13, 14 Some students may not be able to easily make the jump from how three sides determine a triangle to the congruence of two triangles because of corresponding congruent sides. [Ask] "If you made one triangle with these three sides and then made another one, would the triangles be congruent?" [They would, because they're the same.] Solving these exercises requires none of the congruence shortcuts that are to come later in the chapter. Remind students that the vertices of congruent polygons should be labeled in corresponding order, as mentioned in Lesson 1.4.

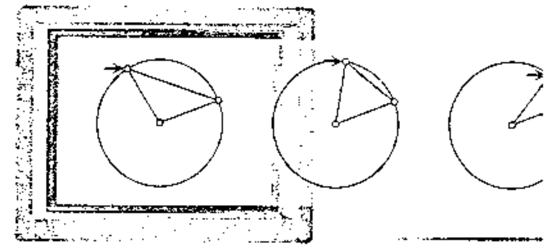
17. possible answer:



12. Study the triangles in the software constructions below. Each triangle has one vertex at the center of the circle, and two vertices on the circle.

a. Are the triangles all isosceles? Write a paragraph proof explaining why or why not.

b. If the vertex at the center of the first circle has an angle measure of 60°, find the measures of the other two angles in that triangle. **60°**



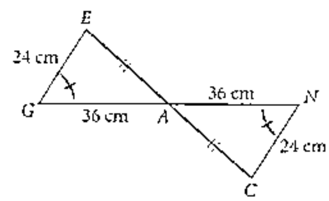
keymath.com/DG

[▸ For an interactive version of this sketch, see the Dynamic Geometry Exploration Triangles in a Circle at www.keymath.com/DG. ◂]

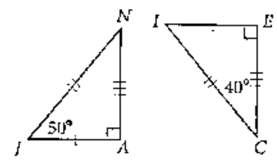
Review

3.6 In Exercises 13 and 14, complete the statement of congruence from the information given. Remember to write the statement so that corresponding parts are in order.

13. $\triangle GEA \cong \triangle$? **NCA**

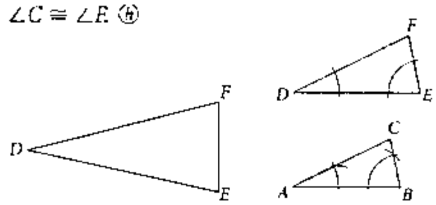


14. $\triangle JAN \cong \triangle$? **IEC**

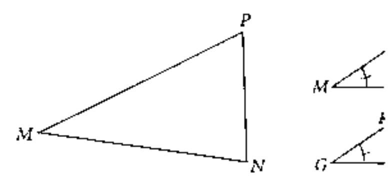


3.6 In Exercises 15 and 16, use a compass or patty paper, and a straightedge, to construct a triangle that is not congruent to the given triangle, but has the given parts congruent. The symbol $\ncong$ means "not congruent to."

15. *Construction* Construct $\triangle ABC \ncong \triangle DEF$ with $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$. ⓗ



16. *Construction* Construct $\triangle GHK \ncong \triangle MN$ with $\overline{HK} \cong \overline{NP}$, $\overline{GH} \cong \overline{MN}$, and $\angle G \cong \angle M$. ⓗ



3.1 **17.** *Construction* With a straightedge and patty paper, construct an angle that measures 105°.

UYAS 3 In Exercises 18–21, determine whether each pair of lines through the points below is parallel, perpendicular, or neither.

$A(1, 3)$ $B(6, 0)$ $C(4, 3)$ $D(1, -2)$ $E(-3, 8)$ $F(-4, 1)$ $G(-1, 6)$ $H(4, -4)$

18. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ ⓗ **perpendicular**

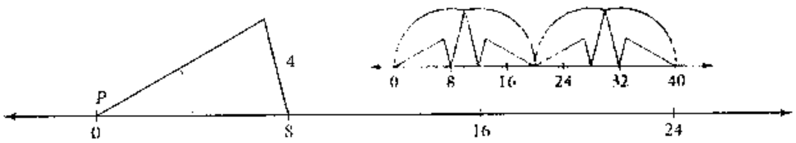
19. $\overleftrightarrow{FG}$ and $\overleftrightarrow{CD}$ **parallel**

20. $\overleftrightarrow{AD}$ and $\overleftrightarrow{CH}$ **parallel**

21. $\overleftrightarrow{DE}$ and $\overleftrightarrow{GH}$ **neither**

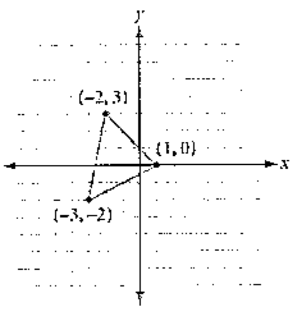
22. Using the preceding coordinate points, is $FGCD$ a trapezoid, a parallelogram, or neither? parallelogram

23. Picture the isosceles triangle below toppling side over side to the right along the line. Copy the triangle and line onto your paper, then construct the path of point P through two cycles. Where on the number line will the vertex point land? 40



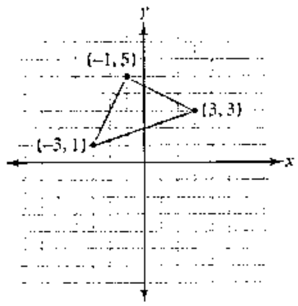
For Exercises 24 and 25, use the ordered pair rule shown to relocate each of the vertices of the given triangle. Connect the three new points to create a new triangle. Is the new triangle congruent to the original one? Describe how the new triangle has changed position from the original.

24. $(x, y) \rightarrow (x + 5, y - 3)$



New: (6, −3), (2, −5), (3, 0). Triangles are congruent.

25. $(x, y) \rightarrow (x, -y)$



New: (3, −3), (−3, −1), (−1, −5). Triangles are congruent.

IMPROVING YOUR **REASONING** SKILLS

Hundreds Puzzle

Fill in the blanks of each equation below. All nine digits—1 through 9—must be used, in order! You may use any combination of signs for the four basic operations (+, −, ·, ÷), parentheses, decimal points, exponents, factorial signs, and square root symbols, and you may place the digits next to each other to create two-digit or three-digit numbers.

Example: $1 + 2(3 + 4.5) + 67 + 8 + 9 = 100$

1. $1 + 2 + 3 - 4 + 5 + 6 + \underline{?} + 9 = 100$
2. $1 + 2 + 3 + 4 + 5 + \underline{?} = 100$
3. $1 + 2 + 3 \cdot 4 \cdot 5 \div 6 + \underline{?} = 100$
4. $(-1 - \underline{?}) \div 5 + 6 + 7 + 89 = 100$
5. $1 + 23 - 4 + \underline{?} + 9 = 100$

Exercise 22 Students may also notice it is a rectangle or even a square.

Exercise 23 Students need to extend the number line past 40.

Exercises 24, 25 [Ask] "If the triangles are indeed congruent after being repositioned, describe how each second triangle was created from the given triangle using the word *reflection, slide,* or *rotation.*" [24. slide right 5, down 3; 25. reflection across x-axis]

EXTENSIONS

A. Pose this problem: "Someone claims that it is possible to divide any triangle into two isosceles triangles and a kite." Have students try this with geometry software or other tools. [Ask] "Do you agree?" If not, ask students to give a counterexample. If so, have them describe their method and explain their results.

B. Use Take Another Look activity 2 or 3 on page 255.

IMPROVING **REASONING** SKILLS

If students are having difficulty with any of these equations, ask which digits are used in the blank and what number the digits must equal when combined.

1. $1 + 2 + 3 - 4 + 5 + 6 + 78 + 9$
2. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \cdot 9$
3. $1 + 2 + 3 \cdot 4 \cdot 5 \div 6 + 78 + 9$
4. $(-1 - 2 - 3 - 4) \div 5 + 6 + 7 + 89$
5. $1 + 23 - 4 + 56 + 7 + 8 + 9$

Many other identities using all nine digits in order are possible, including $12 + 34 + 5 \cdot 6 + 7 + 8 + 9 = 100$, $123 - 4 - 5 - 6 - 7 + 8 - 9 = 100$, and $-1 + 2 + 3 + 4 \cdot 5 \cdot 6 - 7 - 8 - 9 = 100$. You might suggest that students continue to search for other unique equations to add to a class list.

PLANNING

LESSON OUTLINE

One day:

15 min Examples

30 min Exercises

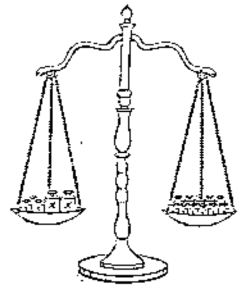
TEACHING

Students see how solving a linear equation is an example of deductive thinking based on algebraic properties.

[ELL] Model using algebraic terms such as *distribute, substitute, simplify,* and *solve* as you do the examples.

Assessing Progress

You can assess students' understanding of solving linear equations and checking the solutions.



Solving Equations

In this chapter you may have already been solving some problems by setting up and solving an equation. An equation is a statement that two expressions are equal. The solution to an equation is the value (or values) of the variable that makes the equation true. The solution to the equation $2x + 3 = 11$ is $x = 4$. You can check this by substituting 4 for x to see that $2(4) + 3 = 11$ is a true equation.

While there are many properties of real numbers and properties of equality, here are some of the main properties that help you solve equations.

Distributive Property

$a(b + c) = a \cdot b + a \cdot c$

This property allows you to simplify equations by separating the terms within parentheses.

$3(5x - 7) = 15x - 21$

Combining like terms

$ax + bx = (a + b)x$

This process, based on the distributive property, allows you to simplify one side of an equation by adding the coefficients of expressions with the same variable.

$-4y + 9y = (-4 + 9)y = 5y$

Properties of Equality

Given $a = b$, for any number c,

Addition property

$a + c = b + c$

Subtraction property

$a - c = b - c$

Multiplication property

$ac = bc$

Division property

$\frac{a}{c} = \frac{b}{c}$ (provided $c \neq 0$)

These properties allow you to perform the same operation on both sides of an equation.

Substitution property

If $a = b$, then a can be replaced with b in any equation.

This property allows you to check your solution to an equation by replacing each variable with its value. Substitution is also used to solve some equations and in writing proofs.

LESSON OBJECTIVES

- Review properties of real numbers and properties of equality
- Consider properties and techniques behind the steps of solving and checking linear equations
- Review the definition of a solution to an equation and steps in solving linear equations

NCTM STANDARDS

CONTENT	PROCESS
✓ Number	✓ Problem Solving
✓ Algebra	✓ Reasoning
✓ Geometry	Communication
Measurement	Connections
Data/Probability	Representation

EXAMPLE A | Solve $4x + 8 = -4(2x - 7) + 4$.

▸ Solution

$4x + 8 = -4(2x - 7) + 2x$	The original equation.
$4x + 8 = -8x + 28 + 2x$	Distribute.
$4x + 8 = -6x + 28$	Combine like terms.
$10x + 8 = 28$	Add 6x to both sides.
$10x = 20$	Subtract 8 from both sides.
$x = 2$	Divide both sides by 10.

The solution is $x = 2$.

Check that the solution makes the original equation true.

$4(2) + 8 \stackrel{?}{=} -4[2(2) - 7] + 2(2)$	Substitute 2 for x.
$4(2) + 8 \stackrel{?}{=} -4[-3] + 4$	Simplify following the order of operations.
$8 + 8 \stackrel{?}{=} 12 + 4$	
$16 = 16$	The solution checks.

When an equation contains fractions or rational expressions, it is sometimes easiest to "clear" them by multiplying both sides of the equation by a common denominator.

EXAMPLE B | Solve $\frac{x}{2} = \frac{3x}{5} - \frac{1}{4}$.

▸ Solution

The denominators are 2, 5, and 4. The least common denominator is 20.

$\frac{x}{2} = \frac{3x}{5} - \frac{1}{4}$	The original equation.
$20\left(\frac{x}{2}\right) = 20\left(\frac{3x}{5} - \frac{1}{4}\right)$	Multiply both sides by 20.
$\frac{20x}{2} = \frac{60x}{5} - \frac{20}{4}$	Distribute.
$10x = 12x - 5$	Reduce the fractions. Now solve as you did in Example A.
$-2x = -5$	Subtract 12x from both sides.
$x = 2.5$	Divide both sides by −2.

Check the solution.

$\frac{2.5}{2} \stackrel{?}{=} \frac{3(2.5)}{5} - \frac{1}{4}$	Substitute 2.5 for x.
$1.25 \stackrel{?}{=} 1.5 - 0.25$	Simplify following the order of operations.
$1.25 = 1.25$	

IGNING HOMEWORK

ntial	1–13
ormance ssment	17
p	12–16

▸ Helping with the Exercises

[ELL] Students with strong algebra backgrounds may find Exercises 1–9 fairly easy. Encourage them to help other group members who are struggling. You might work through 15 and 16 as a class.

Exercise 14 If you avoid cross multiplication because students often misapply it and rarely understand it, this exercise gives an opportunity for students to explain it in terms of number properties.

▸ EXAMPLE A

Although students may be able to solve this equation without showing all the steps, emphasize that the goal here is to think about the reasons behind the process, not just to practice the process.

Emphasize that the question marks over the equal signs are needed for the check to be valid. To prove that you found a solution, you must demonstrate (not assume) that the two sides of the equation are equal after substitution.

▸ EXAMPLE B

To encourage students to engage in creative problem solving while becoming familiar with the properties, you might encourage groups to solve the equation, giving reasons, in an order they think no other group will use.

[Ask] "What does the Solution heading in each example mean?" Elicit the fact that the word *solution* can mean either a number that, when substituted, gives a true equation or the process of solving to get such a number. You might also ask what other meanings *solution* has in English. [It can also refer to a combination of substances, as in chemistry.]

Closing the Lesson

Deductive reasoning appears in all of mathematics, not only in geometry. Steps in solving an algebraic equation have logical reasons in the properties of numbers, especially the distributive property; combining like terms; and the properties of equality. The substitution property is used for proving that you have indeed found a solution.

BUILDING UNDERSTANDING

In the exercises students check solutions to equations, practice solving linear equations, and apply the properties of numbers.

14b. $x = \frac{9}{4}$; the two methods produce identical results. Multiplying by the lowest common denominator (which is comprised of the factors of both denominators and [illegible] common factors), which clears the denominators on either side, is the same as simply multiplying each numerator by the opposite denominator, or cross multiplying. Algebraically you could show that the two methods are equivalent as follows.

$$\frac{a}{b} = \frac{c}{d}$$

$$bd\left(\frac{a}{b}\right) = bd\left(\frac{c}{d}\right)$$

$$\frac{abd}{b} = \frac{bcd}{d}$$

$$ad = bc$$

The method of "clearing fractions" results in the method of "cross multiplying."

Exercise 15 You might have students graph the equations $y = 2(3x + 1)$ and $y = 6x + 3$. A solution to the original equation would be represented by a point of intersection. But the lines are parallel, so there's no solution. This exercise previews proof by contradiction from Chapter 13. The process of finding a solution is a proof of a statement in the form *If x is a solution to the equation, then x =* ___. When the assumption *If x is a solution to the equation* leads to a contradiction, as here, then the assumption must be false, and the equation has no solution.

Exercise 16 Just as some linear equations, such as the one in Exercise 15, have no solutions, some have infinitely many solutions. The expressions on the two sides of the equation are logically equivalent, meaning they have the same value for every real number. If students make graphs as in Exercise 15, the two equations give the same line. That is, all real numbers are solutions to the original equation. Indeed, 0 is a solution, but Camella is mistaken in saying that *the* solution is 0. You might ask students if they can think of examples outside of mathematics in which people claim there's only one answer but there are many.

[illegible]

Exercises

In Exercises 1–2, state whether each equation is true or false.

1. $2(4 + 5) = 13$

2. $2 + [3(-4) - 4] = 2(-4 - 3)$

In Exercises 3–5, determine whether the value given for the variable is a solution to the equation.

3. $x - 8 = 2;\ x = 6$

4. $4(3y - 1) = -40;\ y = -3$

5. $\frac{3}{4}n - \frac{1}{2} = \frac{1}{8};\ n = \frac{1}{4}$

In Exercises 6–13, solve the equation and check your solution.

6. $6x - 3 = 39$

7. $3y - 7 = 5y + 1$

8. $6x - 4(3x + 8) = 16$

9. $7 - 3(2x - 5) = 1 - x$

10. $5(n - 2) - 14n = -3n - (5 - 4n)$

11. $\frac{1}{6} = \frac{3}{10} - \frac{1}{15}x$

12. $\frac{4}{t} - \frac{1}{6} = \frac{1}{t}$

13. $\frac{3n}{4} - \frac{1}{3} = \frac{2n - 1}{6}$

14. A proportion is a statement of equality between two ratios. For example, $\frac{3}{4} = \frac{x}{3}$ is a proportion.
   a. Solve the proportion above by clearing the fractions, as in Example B.
   b. You may have previously learned that you can solve a proportion by "cross multiplying." If so, use this method to solve the proportion above, and compare this method to the one you used in part a.

15. Try solving $2(3x + 1) = 6x + 3$. Explain what happens. What is the solution?

16. Below is Camella's work solving an equation. Camella says, "The solution is zero." Is she correct? Explain.

$2x + 2(x - 1) = 4(x - 3) + 10$	Original equation.
$2x + 2x - 2 = 4x - 12 + 10$	Distribute.
$4x - 2 = 4x - 2$	Combine like terms.
$4x = 4x$	Add 2 to both sides.
$0 = 0$	Subtract $4x$ from both sides.

17. A golden triangle is an isosceles triangle with many special properties. One of them is the measure of either of its base angles is twice the measure of its vertex angle. Sketch a golden triangle with variables representing the angles. Apply previous conjectures to write an equation. Then solve it to determine the measure of each angle.

The diagonals of a regular pentagon form many different golden triangles.

LESSON

4.3

's are plentiful, thinkers

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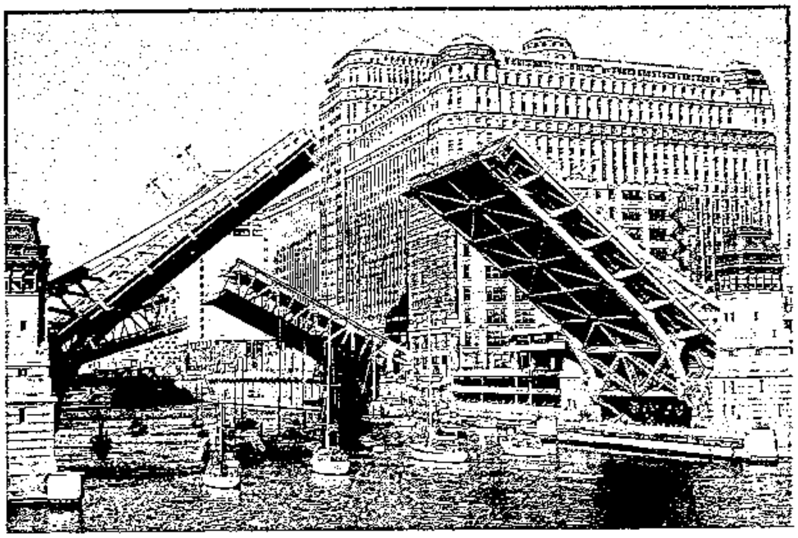
T MARTINEAU

Triangle Inequalities

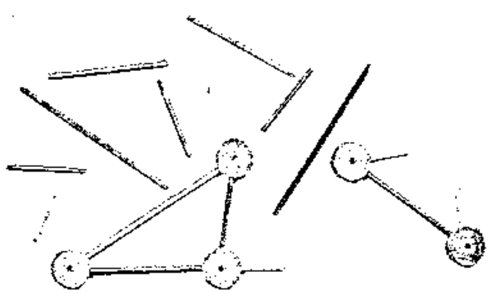
How long must each side of this drawbridge be so that the bridge spans the river when both sides come down?



'bridges over the
igo River in
igo, Illinois

The sum of the lengths of the two parts of the drawbridge must be equal to or greater than the distance across the waterway. Triangles have similar requirements.

You can build a triangle using one blue rod, one green rod, and one red rod. Could you still build a triangle if you used a yellow rod instead of the green rod? Why or why not? Could you form a triangle with two yellow rods and one green rod? What if you used two green rods and one yellow rod?



How can you determine which sets of three rods can be arranged into triangles and which can't? How do the measures of the angles in the triangle relate to the lengths of the rods? How is measure of the exterior angle formed by the yellow and blue rods in the triangle above related to the measures of the angles inside the triangle? In this lesson you will investigate these questions.

M STANDARDS

NTENT	PROCESS	
Number		Problem Solving
Algebra	✓	Reasoning
Geometry	✓	Communication
Measurement	✓	Connections
Data/Probability		Representation

LESSON OBJECTIVES

- Investigate inequalities among sides and angles in triangles
- Discover the Exterior Angle Conjecture
- Practice construction skills
- Develop reasoning skills

LESSON

4.3

PLANNING

LESSON OUTLINE

One day:

25 min Investigation

10 min Sharing

5 min Closing

5 min Exercises

MATERIALS

- construction tools
- protractors
- sticks or uncooked spaghetti, *optional*
- Triangle Exterior Angle Conjecture (W), *optional*
- Sketchpad activity Triangle Inequalities, *optional*

TEACHING

This lesson concerns three properties of triangles: the triangle inequality, the side-angle inequality, and the exterior angle property.

One Step

To combine the investigations, pose this problem: "Draw a horizontal line, which we'll call the base, and mark two points on it fairly close together. Make a triangle that has those two points as two of its vertices. Now, move one of the two points along the base, keeping the opposite side of the triangle fixed in length. Stop the movement at various points and measure all angles and side lengths. Look for patterns and make conjectures." Students might use geometry software for their experimentation, or you might provide sticks or uncooked spaghetti. While circulating, be sure some groups focus on exterior angles, some on relative side and angle measures, and some on what happens when the triangle disappears.